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Chapter 8: Estimating with Confidence

Section 8.3
Estimating a Population Mean

The Practice of Statistics, 4th edition – For AP*

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Chapter 8
Estimating with Confidence

- 8.1 Confidence Intervals: The Basics
- 8.2 Estimating a Population Proportion
- **8.3 Estimating a Population Mean**

Section 8.3
Estimating a Population Mean

Learning Objectives

After this section, you should be able to…

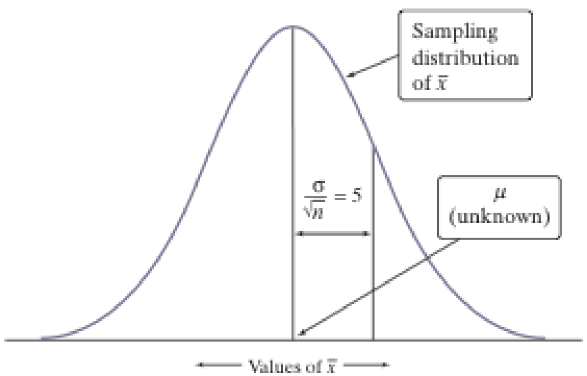
- ✓ CONSTRUCT and INTERPRET a confidence interval for a population mean
- ✓ DETERMINE the sample size required to obtain a level *C* confidence interval for a population mean with a specified margin of error
- ✓ DESCRIBE how the margin of error of a confidence interval changes with the sample size and the level of confidence *C*
- ✓ DETERMINE sample statistics from a confidence interval

The One-Sample *z* Interval for a Population Mean

In Section 8.1, we estimated the "mystery mean" μ (see page 468) by constructing a confidence interval using the sample mean = 240.79.

To calculate a 95% confidence interval for μ , we use the familiar formula:

estimate ± (critical value) • (standard deviation of statistic)



$$
\begin{aligned}
\bar{x} \pm z^* \cdot \frac{\sigma}{\sqrt{n}} &= 240.79 \pm 1.96 \cdot \frac{20}{\sqrt{16}} \\
&= 240.79 \pm 9.8 \\
&= (230.99, 250.59)
\end{aligned}
$$

One-Sample *z* Interval for a Population Mean

Choose an SRS of size *n* from a population having unknown mean μ and known standard deviation σ. As long as the Normal and Independent conditions are met, a level *C* confidence interval for μ is

$$\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}$$

The critical value z^* is found from the standard Normal distribution.

Choosing the Sample Size

The margin of error *ME* of the confidence interval for the population mean μ is

$$z^* \cdot \frac{\sigma}{\sqrt{n}}$$

We determine a sample size for a desired margin of error when estimating a mean in much the same way we did when estimating a proportion.

Choosing Sample Size for a Desired Margin of Error When Estimating μ

To determine the sample size *n* that will yield a level *C* confidence interval for a population mean with a specified margin of error *ME*:

- Get a reasonable value for the population standard deviation σ from an earlier or pilot study.
- Find the critical value z^* from a standard Normal curve for confidence level *C*.
- Set the expression for the margin of error to be less than or equal to *ME* and solve for *n*:

$$z^* \frac{\sigma}{\sqrt{n}} \leq ME$$

Example: How Many Monkeys?

Researchers would like to estimate the mean cholesterol level μ of a particular variety of monkey that is often used in laboratory experiments. They would like their estimate to be within 1 milligram per deciliter (mg/dl) of the true value of μ at a 95% confidence level. A previous study involving this variety of monkey suggests that the standard deviation of cholesterol level is about 5 mg/dl.

- ✓ The critical value for 95% confidence is $z^* = 1.96$.
- ✓ We will use $\sigma = 5$ as our best guess for the standard deviation.

$$1.96\frac{5}{\sqrt{n}} \leq 1$$

Multiply both sides by square root *n* and divide both sides by 1.

$$\frac{(1.96)(5)}{1} \leq \sqrt{n}$$

Square both sides.

$$(1.96 \cdot 5)^2 \leq n$$

$$96.04 \leq n$$

We round up to 97 monkeys to ensure the margin of error is no more than 1 mg/dl at 95% confidence.

Alternate Example: How Much Homework?

- Administrators at your school want to estimate how much time students spend on homework, on average, during a typical week. They want to estimate at the 90% confidence level with a margin of error of at most 30 minutes. A pilot study indicated that the standard deviation of time spent on homework per week is about 154 minutes.

- **Problem:** How many students need to be surveyed to estimate the mean number of minutes spent on homework per week with 90% confidence and a margin of error of at most 30 minutes?

- **Solution:**

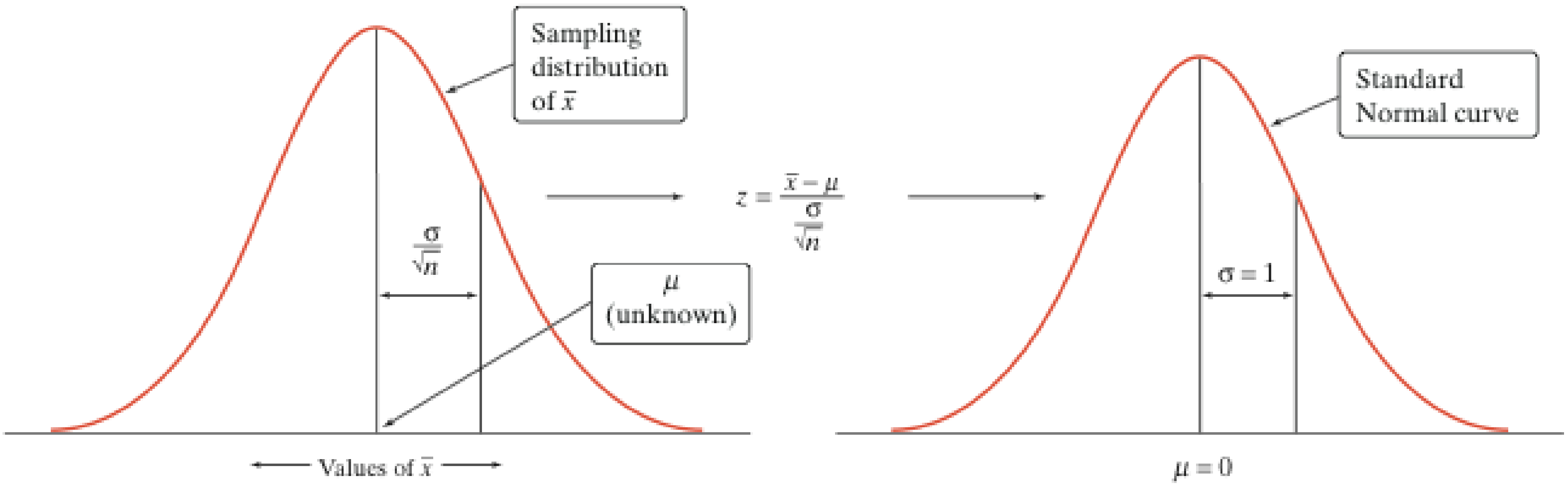
$$1.645\frac{154}{\sqrt{n}} \le 15 \to \left(1.645\frac{154}{15}\right)^2 \le n \to 285.2 \le n$$

The administrators need to survey at least 286 students.

When σ is Unknown: The *t* Distributions

When the sampling distribution of $\bar{x}$ is close to Normal, we can find probabilities involving $\bar{x}$ by standardizing:

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$



When we don't know σ, we can estimate it using the sample standard deviation s_x. What happens when we standardize?

$$?? = \frac{\bar{x} - \mu}{s_x / \sqrt{n}}$$

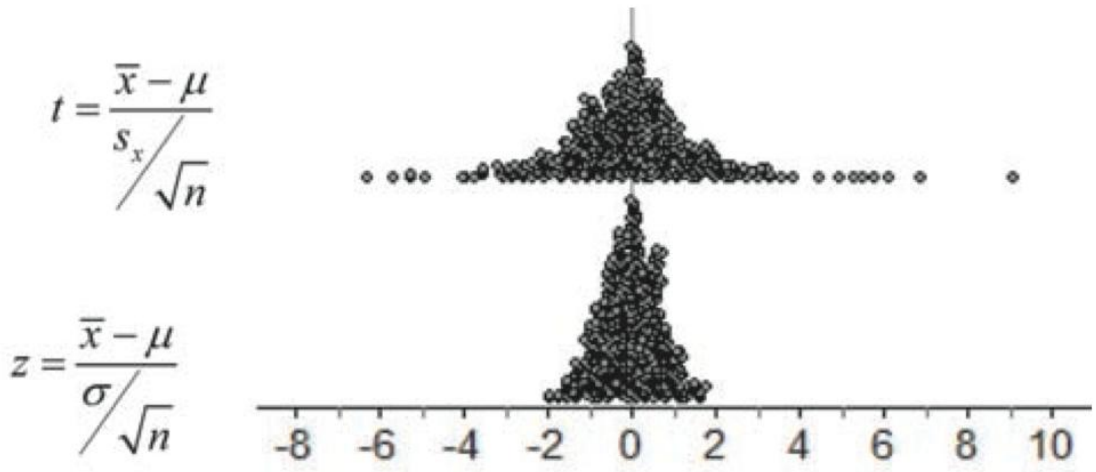
This new statistic does *not* have a Normal distribution!

When σ is Unknown: The *t* Distributions

When we standardize based on the sample standard deviation s_x, our statistic has a new distribution called a ***t* distribution**.

It has a *different shape* than the standard Normal curve:

- It is symmetric with a single peak at 0,
- However, it has much more area in the tails.



Like any standardized statistic, t tells us how far $\bar{x}$ is from its mean μ in standard deviation units.

However, there is a different *t* distribution for each sample size, specified by its **degrees of freedom (df)**.

The t Distributions; Degrees of Freedom

When we perform inference about a population mean μ using a t distribution, the appropriate degrees of freedom are found by subtracting 1 from the sample size n, making df = n - 1. We will write the t distribution with n - 1 degrees of freedom as t_{n-1}.

The t Distributions; Degrees of Freedom

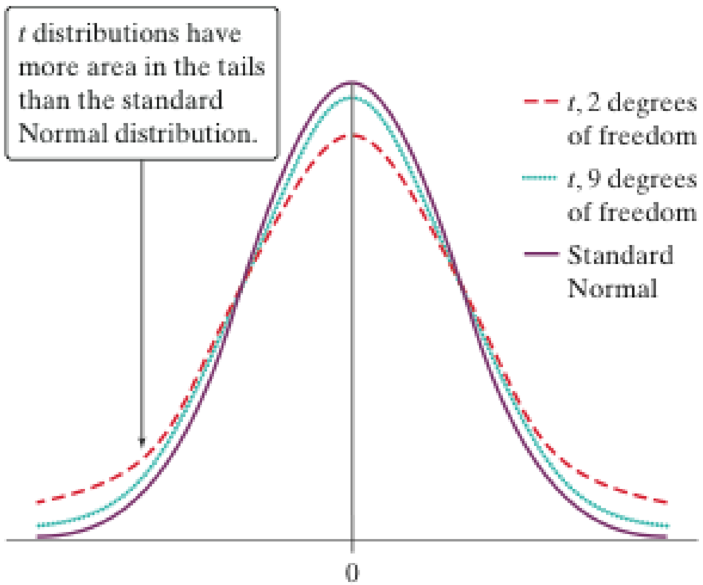
Draw an SRS of size n from a large population that has a Normal distribution with mean μ and standard deviation σ. The statistic

$$t = \frac{\bar{x} - \mu}{s_x / \sqrt{n}}$$

has the ***t* distribution** with **degrees of freedom** df = n – 1. The statistic will have approximately a t_{n-1} distribution as long as the sampling distribution is close to Normal.

The *t* Distributions; Degrees of Freedom

When comparing the density curves of the standard Normal distribution and *t* distributions, several facts are apparent:



✓The density curves of the *t* distributions are similar in shape to the standard Normal curve.

✓The spread of the *t* distributions is a bit greater than that of the standard Normal distribution.

✓The *t* distributions have more probability in the tails and less in the center than does the standard Normal.

✓As the degrees of freedom increase, the *t* density curve approaches the standard Normal curve ever more closely.

We can use Table B in the back of the book to determine critical values *t** for *t* distributions with different degrees of freedom.

Using Table B to Find Critical t^* Values

Suppose you want to construct a 95% confidence interval for the mean μ of a Normal population based on an SRS of size n = 12. What critical t^* should you use?

	Upper-tail probability *p*			
df	**.05**	**.025**	**.02**	**.01**
10	1.812	2.228	2.359	2.764
11	1.796	2.201	2.328	2.718
12	1.782	2.179	2.303	2.681
z*	1.645	1.960	2.054	2.326
	90%	95%	96%	98%
	Confidence level *C*			

In Table B, we consult the row corresponding to df = n – 1 = 11.

We move across that row to the entry that is directly above 95% confidence level.

The desired critical value is t^* = 2.201.

Alternate Example – Finding t*

- **Problem:** Suppose you wanted to construct a 90% confidence interval for the mean of a Normal population based on an SRS of size 10. What critical value *t** should you use?

  Solution: Using the line for df = 10 – 1 = 9 and the column with a tail probability of 0.05, the desired critical value is *t** = 1.833.

Constructing a Confidence Interval for μ

When the conditions for inference are satisfied, the sampling distribution for $\bar{x}$ has roughly a Normal distribution. Because we don't know σ, we estimate it by the sample standard deviation s_x.

Definition:

The **standard error of the sample mean** $\bar{x}$ is $\frac{s_x}{\sqrt{n}}$, where s_x is the sample standard deviation. It describes how far $\bar{x}$ will be from μ, on average, in repeated SRSs of size n.

To construct a confidence interval for μ,

✓ Replace the standard deviation of $\bar{x}$ by its standard error in the formula for the one-sample z interval for a population mean.

✓ Use critical values from the t distribution with n - 1 degrees of freedom in place of the z critical values. That is,

$$\text{statistic} \pm (\text{critical value}) \cdot (\text{standard deviation of statistic})$$

$$= \bar{x} \pm t^* \frac{s_x}{\sqrt{n}}$$

One-Sample *t* Interval for a Population Mean

The **one-sample *t* interval for a population mean** is similar in both reasoning and computational detail to the one-sample *z* interval for a population proportion. As before, we have to verify three important conditions before we estimate a population mean.

Conditions for Inference about a Population Mean

- **Random**: The data come from a random sample of size *n* from the population of interest or a randomized experiment.
- **Normal**: The population has a Normal distribution or the sample size is large ($n \geq 30$).
- **Independent**: The method for calculating a confidence interval assumes that individual observations are independent. To keep the calculations reasonably accurate when we sample without replacement from a finite population, we should check the *10% condition*: verify that the sample size is no more than 1/10 of the population size.

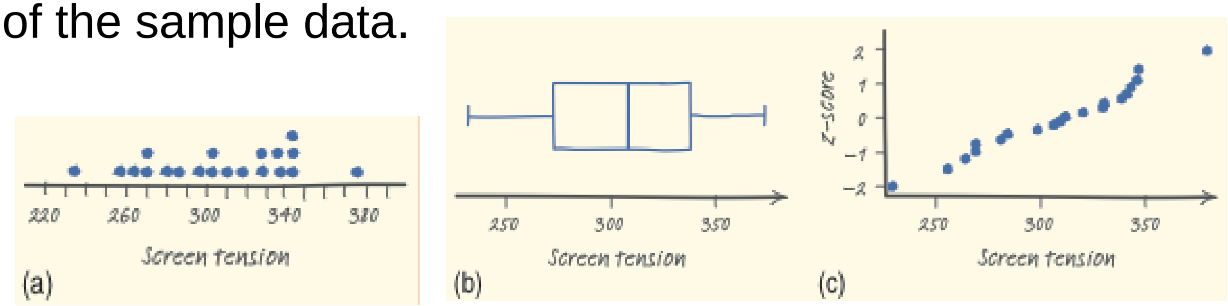
Example: Video Screen Tension

Read the Example on page 508. **STATE:** **We want to estimate the true mean tension μ of all the video terminals produced this day at a 90% confidence level.**

PLAN: If the conditions are met, we can use a one-sample *t* interval to estimate μ.

Random: We are told that the data come from a random sample of 20 screens from the population of all screens produced that day.

Normal: Since the sample size is small ($n < 30$), we must check whether it's reasonable to believe that the population distribution is Normal. Examine the distribution of the sample data.



These graphs give no reason to doubt the Normality of the population

Independent: Because we are sampling without replacement, we must check the 10% condition: we must assume that at least 10(20) = 200 video terminals were produced this day.

Example: Video Screen Tension

STEP 4

Read the Example on page 508. **We want to estimate the true mean tension μ of all the video terminals produced this day at a 90% confidence level.**

DO: Using our calculator, we find that the mean and standard deviation of the 20 screens in the sample are:

$$\bar{x} = 306.32 \text{ mV} \qquad \text{and} \qquad s_x = 36.21 \text{ mV}$$

Upper-tail probability *p*

df	**.10**	**.05**	**.025**
18	1.130	1.734	2.101
19	1.328	1.729	2.093
20	1.325	1.725	2.086
	90%	95%	96%

Confidence level *C*

Since $n = 20$, we use the t distribution with df = 19 to find the critical value.

From Table B, we find $t^* = 1.729$.

Therefore, the 90% confidence interval for μ is:

$$\bar{x} \pm t^* \frac{s_x}{\sqrt{n}} = 306.32 \pm 1.729 \frac{36.21}{\sqrt{20}}$$

$$= 306.32 \pm 14$$

$$= (292.32,\ 320.32)$$

CONCLUDE: We are 90% confident that the interval from 292.32 to 320.32 mV captures the true mean tension in the entire batch of video terminals produced that day.

Alternate Example: Can you spare a square?

- As part of their final project in AP Statistics, Christina and Rachel randomly selected 18 rolls of a generic brand of toilet paper to measure how well this brand could absorb water. To do this, they poured 1/4 cup of water onto a hard surface and counted how many squares it took to completely absorb the water. Here are the results from their 18 rolls:

29	20	25	29	21	24	27	25	24
29	24	27	28	21	25	26	22	23

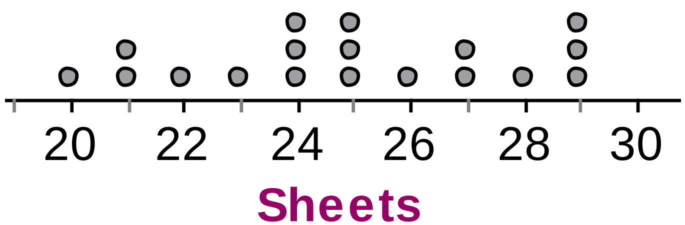
- **STATE:** We want to estimate = the mean number of squares of generic toilet paper needed to absorb 1/4 cup of water with 99% confidence.

PLAN: If the conditions are met, we can use a one-sample *t* interval to estimate μ.

Random: We are told that the data come from a random sample of 20 screens from the population of all screens produced that day.

Normal: Since the sample size is small ($n < 30$), we must check whether it's reasonable to believe that the population distribution is Normal. Examine the distribution of the sample data.

These graphs give no reason to doubt the Normality of the population



Independent: Because we are sampling without replacement, we must check the 10% condition: we must assume that at least 10(20) = 200 video terminals were produced this day.

Alternate Example: Can you spare a square?

- **DO:** The sample mean for these data is = 24.94 and the sample standard deviation is = 2.86. Since there are 18 – 1 = 17 degrees of freedom and we want 99% confidence, we will use a critical value of *t** = 2.898.

$$\bar{x} \pm t^* \frac{s_x}{\sqrt{n}} = 24.94 \pm 2.898 \frac{2.86}{\sqrt{18}} = 24.94 \pm 1.95 = (22.99, 26.89)$$

Conclude: We are 99% confident that the interval from 22.99 squares to 26.89 squares captures the true mean number of squares of generic toilet paper needed to absorb 1/4 cup of water.

Alternate Example: How much homework?

- The principal at a large high school claims that students spend at least 10 hours per week doing homework on average. To investigate this claim, an AP Statistics class selected a random sample of 250 students from their school and asked them how many long they spent doing homework during the last week. The sample mean was 10.2 hours and the sample standard deviation was 4.2 hours.
- **Problem:**
- (a) Construct and interpret a 95% confidence interval for the mean time spent doing homework in the last week for students at this school.
- (b) Based on your interval in part (a), what can you conclude about the principal's claim?

Solution:

(a) STATE: We want to estimate μ = the mean time spent doing homework in the last week for students at this school with 95% confidence.

Alternate Example: How much homework?

Plan: We will construct a one-sample *t* interval provided the following conditions are met:

- Random: The students were randomly selected.
- Normal: The sample size is large (n = 250), so we are safe using *t*-procedures.
- Independent: Because we are sampling without replacement, we must check the 10% condition. It is reasonable to believe there are more than 10(250) = 2500 students at this large high school.

Do: Because there are 250 – 1 = 249 degrees of freedom and we want 95% confidence, we will use the *t*-table and a conservative degrees of freedom of 100 to get a critical value of t^* = 1.984.

$$\bar{x} \pm t^* \frac{s_x}{\sqrt{n}} = 10.2 \pm 1.984 \frac{4.2}{\sqrt{250}} = 10.2 \pm 0.53 = (9.67, 10.73)$$

Conclude: We are 95% confident that the interval from 9.67 hours to 10.73 hours captures the true mean number of hours of homework that students at this school did in the last week.

(b) Since the interval of plausible values for μ includes values less than 10, the interval does not provide convincing evidence to support the principal's claim that students spend at least 10 hours on homework per week, on average.

Using *t* Procedures Wisely

The stated confidence level of a one-sample *t* interval for μ is exactly correct when the population distribution is exactly Normal. No population of real data is exactly Normal. The usefulness of the *t* procedures in practice therefore depends on how strongly they are affected by lack of Normality.

Definition:

An inference procedure is called **robust** if the probability calculations involved in the procedure remain fairly accurate when a condition for using the procedures is violated.

Fortunately, the *t* procedures are quite robust against non-Normality of the population except when outliers or strong skewness are present. Larger samples improve the accuracy of critical values from the *t* distributions when the population is not Normal.

Using *t* Procedures Wisely

Except in the case of small samples, the condition that the data come from a random sample or randomized experiment is more important than the condition that the population distribution is Normal. Here are practical guidelines for the Normal condition when performing inference about a population mean.

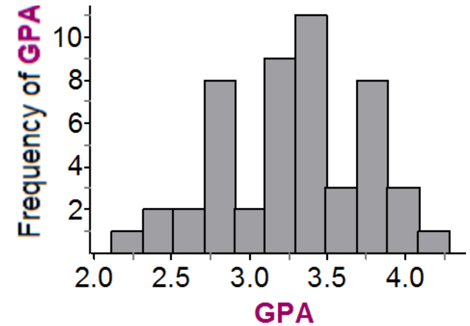
Using One-Sample *t* Procedures: The Normal Condition

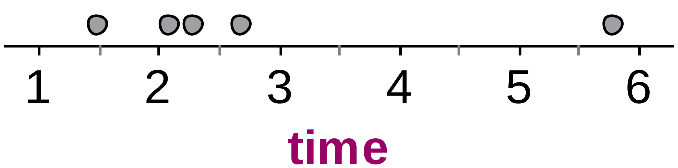
- *Sample size less than 15*: Use *t* procedures if the data appear close to Normal (roughly symmetric, single peak, no outliers). If the data are clearly skewed or if outliers are present, do not use *t*.
- *Sample size at least 15*: The *t* procedures can be used except in the presence of outliers or strong skewness.
- *Large samples*: The *t* procedures can be used even for clearly skewed distributions when the sample is large, roughly $n \geq 30$.

Alternate Example: GPA, coffee and SAT scores?

Problem: Determine whether we can safely use a one-sample *t* interval to estimate the population mean in each of the following settings.

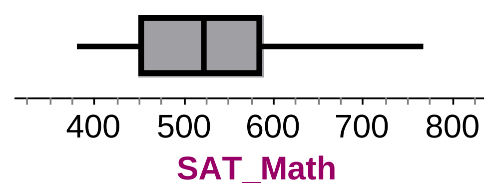
(a) To estimate the average GPA of students at your school, you randomly select 50 students from classes you take. Here is a histogram of their GPAs.





(b) The dotplot shows the amount of time it took to order and receive a regular coffee in 5 visits to a local coffee shop.

(c) The boxplot below shows the SAT math score for a random sample of 20 students at your high school.



Solution:

(a) Since the sample of 50 students was only from your classes and not from all students at your school, we should not use a *t* interval to generalize about the mean GPA for all students at the school.

(b) Since the sample size is small and there is a possible outlier, we should not use a *t* interval.

(c) Since the distribution is only moderately skewed and the sample size is larger than 15, it is safe to use a *t* interval.

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Section 8.3
Estimating a Population Mean

Summary

In this section, we learned that…

- ✓ **Confidence intervals for the mean μ of a Normal population** are based on the sample mean of an SRS.
- ✓ If we somehow know σ, we use the *z* critical value and the standard Normal distribution to help calculate confidence intervals.
- ✓ The sample size needed to obtain a confidence interval with approximate margin of error ME for a population mean involves solving

$$z^* \frac{\sigma}{\sqrt{n}} \leq ME$$

- ✓ In practice, we usually don't know σ. Replace the standard deviation of the sampling distribution with the **standard error** and use the *t* distribution with $n - 1$ **degrees of freedom (df)**.

Section 8.3
Estimating a Population Mean

Summary

- ✓ There is a *t* distribution for every positive degrees of freedom. All are symmetric distributions similar in shape to the standard Normal distribution. The *t* distribution approaches the standard Normal distribution as the number of degrees of freedom increases.
- ✓ A level C confidence interval for the mean μ is given by the one-sample *t* interval

$$\bar{x} \pm t^* \frac{s_x}{\sqrt{n}}$$

- ✓ This inference procedure is approximately correct when these conditions are met: Random, Normal, Independent.
- ✓ The *t* procedures are relatively robust when the population is non-Normal, especially for larger sample sizes. The *t* procedures are not robust against outliers, however.

Looking Ahead…

In the next Chapter…

We'll learn how to test a claim about a population.

We'll learn about

- ✓ **Significance Tests: The Basics**
- ✓ **Tests about a Population Proportion**
- ✓ **Tests about a Population Mean**